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THE GENERAL POSITION PROBLEM ON KNESER GRAPHS AND ON SOME GRAPH OPERATIONS

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Abstract

A vertex subset S of a graph G is a general position set of G if no vertex of S lies on a geodesic between two other vertices of S. The cardinality of a largest general position set of G is the general position number (gp-number) $\mathrm{gp}(G)$ of G. The gp-number is determined for some families of Kneser graphs, in particular for $K(n,2)$, $n \geq 4$, and $K(n,3)$, $n \geq 9$. A sharp lower bound on the gp-number is proved for Cartesian products of graphs. The gp-number is also determined for joins of graphs, coronas over graphs, and line graphs of complete graphs.

Keywords: general position set, Kneser graphs, Cartesian product of graphs, corona over graphs, line graphs.

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1. Introduction

A general position problem in graph theory is to find a largest set of vertices that are in a general position. More precisely, if $G = (V(G), E(G))$ is a graph, then $S \subseteq V(G)$ is a *general position set* if for any triple of pairwise different vertices $u, v, w \in S$ we have $d_G(u,v) \neq d_G(u,w) + d_G(w,v)$, where d_G is the standard shortest path distance function in the graph G. A set S is called a *gp-set* of G if S has the largest cardinality among the general position sets of G. The *general position number* (*gp-number* for short) $\mathrm{gp}(G)$ of G is the cardinality of a gp-set of G.

This concept was introduced—under the present name—in [14] in part motivated by the Dudeney's 1917 no-three-in-line problem [5] (see [12, 16, 20] for recent related results) and by a corresponding problem in discrete geometry known as the general position subset selection problem [7, 19]. Independently geodetic irredundant sets were earlier introduced in [21], a concept which is equivalent to the general position sets.

We will use $n(G)$ to denote the order of G. In [21] graphs G with $\mathrm{gp}(G) \in \{2, n(G)-1, n(G)\}$ were classified and some other results presented. Then, in [14], several general bounds on the gp-number were presented, proved that set of simplicial vertices of a block graph form its gp-set, and proved that the problem is NP-complete in general. The gp-number of a large class of subgraphs of the infinite grid graph and of the infinite diagonal grid has been determined in [15]. In the paper [1] a formula for the gp-number of graphs of diameter 2 was given which in particular implies that $\mathrm{gp}(G)$ of a cograph G can be determined in polynomial time. Moreover, a formula for the gp-number of the complement of a bipartite graph was also deduced. The main result of [1] gives a characterization of general position sets (see Theorem 1.2 below). The general position problem has also been connected with the so-called strong resolving graphs [11].

We proceed as follows. In the rest of this section further definitions are given, and known results needed are stated. In Section 2 the gp-number is determined for some families of Kneser graphs. In particular, if $n \geq 7$, then $\mathrm{gp}(K(n,2)) = n-1$ and if $n \geq 9$, then $\mathrm{gp}(K(n,3)) = \binom{n-1}{2}$. In the subsequent section the gp-number of Cartesian products is bounded from below. The bound is proved to be sharp on the Cartesian product of two complete graphs. We conclude the paper with Section 4 in which the gp-number is determined for joins of graphs, coronas over graphs, and line graphs of complete graphs, where the first two results are stated as functions of the corresponding invariants of factor graphs.

For a positive integer n let $[n] = \{1, \ldots, n\}$. Graphs in this paper are finite, undirected, and simple. The maximum distance between all pairs of vertices of G is the *diameter*, $\mathrm{diam}(G)$ of G. An u,v-path of length $d_G(u,v)$ is called an u,v-*geodesic*. The *interval* $I_G(u,v)$ between vertices u and v of a graph G is the set of vertices x such that there exists a u,v-geodesic which contains x. A subgraph H of G is *convex* if for every $u,v \in V(H)$, all the vertices from $I_G(u,v)$ belong to $V(H)$.

The size of a largest complete subgraph of a graph G and the size of its largest independent set are denoted by $\omega(G)$ and $\alpha(G)$, respectively. The complement of a graph G will be denoted with $\overline{G}$ and the subgraph of G induced by $S \subseteq V(G)$ with $G[S]$. Let $\eta(G)$ denote the maximum order of an induced complete multipartite subgraph of $\overline{G}$. We will use the following result.

Theorem 1.1. [1, Theorem 4.1] *If* $\mathrm{diam}(G) = 2$, *then* $\mathrm{gp}(G) = \max\{\omega(G), \eta(G)\}$.

To complete the introduction we recall a characterization of general position sets from [1], for which some preparation is required. If G is a connected graph, $S \subseteq V(G)$, and $\mathcal{P} = \{S_1, \ldots, S_p\}$ a partition of S, then $\mathcal{P}$ is *distance-constant* (named "distance-regular" in [9, p. 331]) if for any $i,j \in [p]$, $i \neq j$, the distance $d_G(u,v)$, where $u \in S_i$ and $v \in S_j$, is independent of the selection of u and v. This distance is then the distance $d_G(S_i, S_j)$ between the parts S_i and S_j. A distance-constant partition $\mathcal{P}$ is *in-transitive* if $d_G(S_i, S_k) \neq d_G(S_i, S_j) + d_G(S_j, S_k)$ holds for arbitrary pairwise different $i,j,k \in [p]$. Then we have:

Theorem 1.2. [1, Theorem 3.1] *Let G be a connected graph. Then $S \subseteq V(G)$ is a general position set if and only if the components of $G[S]$ are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of S.*

Theorem 1.2 is illustrated in Fig. 1 on the Petersen graph P. It is known (cf. [14]) that $\mathrm{gp}(P) = 6$, the end-vertices of the red edges form its gp-set. Note that these six vertices induce three (complete subgraphs) K_2, and that the distance between each pair of these complete subgraphs is 2.

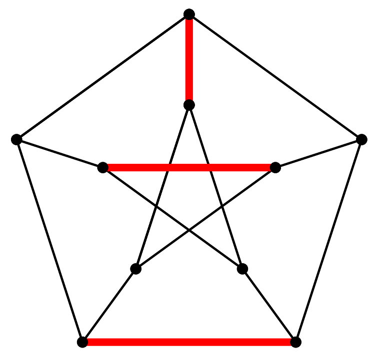

Figure 1. A gp-set of the Petersen graph

2. Kneser graphs

If n and k are positive integers with $n \geq k$, then the *Kneser graph* $K(n, k)$ has as vertices all the k-element subsets of the set $[n]$, vertices being adjacent if the corresponding sets are disjoint. For more on Kneser graph see [2, 3, 17, 22].

In this section we are interested in the gp-number of Kneser graphs, for which the following result will be useful.

Theorem 2.1. [22, Theorem 1] *If* $k \geq 1$ *and* $n \geq 2k + 1$, *then* $\mathrm{diam}(K(n, k)) = \lceil (k - 1)/(n - 2k) \rceil + 1$.

Recall also that the celebrated Erdős-Ko-Rado theorem [6] asserts that if $n \geq 2k$, then $\alpha(K(n, k)) \leq \binom{n-1}{k-1}$, cf. also [13, Theorem 6.4].

In our first result of the section we determine the gp-number of the Kneser graphs $K(n, 2)$ as follows.

Theorem 2.2. *If* $n \geq 4$, *then*

$$\mathrm{gp}(K(n, 2)) = \begin{cases} 6; & 4 \leq n \leq 6, \\ n - 1; & n \geq 7\,. \end{cases}$$

Proof. Since $K(4, 2) = 3K_2$, clearly we have $\mathrm{gp}(K(4, 2)) = 6$. The Kneser graph $K(5, 2)$ is the Petersen graph for which it has been proven in [14] that $\mathrm{gp}(K(5, 2)) = 6$.

We now claim that $\mathrm{gp}(K(n, 2)) \leq n-1$ for every $n \geq 7$, and that $\mathrm{gp}(K(6, 2)) \leq 6$. For this sake let S be an arbitrary general position set of $K(n, 2)$. By Theorem 1.2 the components of $K(n, 2)[S]$ are complete graphs. We distinguish the following cases based on the cardinality of a largest component, say H, of $K(n, 2)[S]$.

Let $n(H) \geq 3$, and assume without loss of generality that $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$ are vertices of H. Then an arbitrary vertex x from $V(K(n,2)) \setminus V(H)$ can have a non-empty intersection with at most two of the vertices $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$. This implies that x is adjacent to at least one vertex of H. It follows that $K(n,2)[S]$ has only one (complete) component, and consequently $|S| \leq \lfloor \frac{n}{2} \rfloor$.

Let $n(H) = 2$. Assume without loss of generality that $V(H) = \{\{1,2\},\{3,4\}\}$. Since no other vertex of S is adjacent with the vertices of K_2, the other vertices of S must be 2-subsets of $[4]$. Hence in this case $|S| \leq 6$.

Let $n(H) = 1$, that is, S is an independent set. Then the Erdős-Ko-Rado theorem implies that $|S| \leq n-1$.

From the above three cases we conclude that $\mathrm{gp}(K(6,2)) \leq 6$, and that $\mathrm{gp}(K(n,2)) \leq n-1$ holds for every $n \geq 7$. It remains to prove that for $n \geq 6$ we can construct large enough general position sets.

Suppose that $n = 6$. Then the six 2-subsets of $[4]$ induce three independent edges, hence $\mathrm{gp}(K(6,2)) \geq 6$. By the above we conclude that $\mathrm{gp}(K(6,2)) = 6$.

Let $n \geq 7$. Then by the above, $\mathrm{gp}(K(n,2)) \leq n-1$. On the other hand, the set $\{\{1,2\},\{1,3\},\ldots,\{1,n\}\}$ is an independent set of $K(n,2)$ of cardinality $n-1$. Since $\mathrm{diam}(K(n,2)) = 2$, Theorem 2.1 implies that this independent set is a general position set, hence we conclude that $\mathrm{gp}(K(n,2)) \geq n-1$.

In summary, if $n \geq 7$, then $\mathrm{gp}(K(n,2)) = n-1$. ■

Theorem 2.3. *Let $n,k \in \mathbb{N}$ and $n \geq 3k-1$. If for all t, where $2 \leq t \leq k$, the inequality $k^t\binom{n-t}{k-t} + t \leq \binom{n-1}{k-1}$ holds, then*

$$\mathrm{gp}(K(n,k)) = \binom{n-1}{k-1}.$$

Proof. Since $n \geq 3k-1$, Theorem 2.1 implies that $\mathrm{diam}(K(n,k)) = 2$.

Let $\mathcal{S}$ be the set of all k-subsets of $[n]$ that contain 1. Clearly, $|\mathcal{S}| = \binom{n-1}{k-1}$ and $\mathcal{S}$ form an independent set of $K(n,k)$. Hence, as $\mathrm{diam}(K(n,k)) = 2$, we infer that $\mathcal{S}$ is a general position set and consequently $\mathrm{gp}(K(n,k)) \geq \binom{n-1}{k-1}$.

Let T be a general position set of $K(n,k)$, and let H be a largest component of $K(n,k)[T]$. By Theorem 1.2 we know that H is a complete subgraph. Let $n(H) = t$. If $t > k$, then every vertex $V(K(n,k)) \setminus V(H)$ must have a neighbor in H. This implies that T is the only component of $K(n,k)[T]$, but then we clearly have $n(H) \leq \binom{n-1}{k-1}$. Hence assume in the rest that $t \leq k$.

If $t = 1$, then $K(n,k)[T]$ is a disjoint union of K_1s and hence $|T| \leq \binom{n-1}{k-1}$ by the Erdős-Ko-Rado theorem.

Suppose now $2 \leq t \leq k$. We wish to determine the upper bound on the number of k-subsets A, such that $A \cap B \neq \emptyset$ holds for all $B \in V(H)$. Such a set A must have at least one element from each of the sets $B \in V(H)$, and since the sets B are pairwise disjoint, there are $\binom{k}{1}^t$ possibilities to select representatives from the sets $B \in V(H)$ that are at the same time elements of A. The remaining $k-t$ elements of A are then selected from a set of cardinality $n-t$. Therefore, there exist at most

$$\underbrace{\binom{k}{1}\binom{k}{1}\cdots\binom{k}{1}}_{t\text{-times}}\binom{n-t}{k-t} = k^t\binom{n-t}{k-t}$$

k-sets A, such that $A \cap B \neq \emptyset$ for all $B \in V(H)$. Hence,

$$|T| \le t + k^t\binom{n-t}{k-t} \le \binom{n-1}{k-1},$$

where the second inequality holds by the theorem's assumption. We conclude that $\mathrm{gp}(K(n,k)) = \binom{n-1}{k-1}$. ■

For the Kneser graphs $K(n,3)$ we have the following result.

Theorem 2.4. *If* $n \ge 9$, *then* $\mathrm{gp}(K(n,3)) = \binom{n-1}{2}$.

Proof. Let T be a general position set of $K(n,3)$. By Theorem 1.2, every component of $K(n,3)[T]$ is a clique, and let H be a largest such clique. We first prove that $\mathrm{gp}(K(n,3)) \le \binom{n-1}{2}$ (for $n \ge 9$), for which we distinguish the following cases.

Case 1: $n(H) \ge 4$.
Let x be an arbitrary vertex from $T \setminus V(H)$. Since x must contain a vertex from each of the sets from $V(H)$ and the latter sets are pairwise disjoint, x would contain at least four elements. Since this is not possible, T consists of the single clique H. It follows that $n(H) \le \lfloor n/3 \rfloor \le \binom{n-1}{2}$.

Case 2: $n(H) = 3$.
Let $H_1, \ldots, H_\ell$ be the components of $K(n,3)[T]$ of cardinality 3. As the sets (=vertices) from every H_i are pairwise disjoint, we may without loss of generality assume that $H_1 = \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$. This in particular implies that $n \ge 9$. Then each vertex from every H_i contains elements from $[9]$. Suppose now that the pair $\{1,2\}$ appears in some vertex y different from $\{1,2,3\}$. Then y is adjacent to at least one of the vertices $\{4,5,6\}$ and $\{7,8,9\}$. By symmetry it follows that each pair of elements $\{i,j\} \in \binom{[9]}{2}$ appears in at most one vertex from $V(H_1) \cup \cdots \cup V(H_\ell)$. Since there are 36 such pairs it follows that $\ell \le 4$.

First assume that $\ell = 4$. By the argument above, T contains no other vertex but those in $H_1, \ldots, H_4$. Then $|T| = 12 \le \binom{n-1}{2}$ because $n \ge 9$. Let next $\ell = 3$. Then at most three vertices can have non-empty intersection with all the vertices from H_1, H_2, H_3. Again, $|T| \le \binom{n-1}{2}$. Suppose next that $\ell = 2$. Then each of the cliques allows 27 further vertices to belong to T. The list of possible vertices intersects in only 6 vertices that can lie in T besides the vertices from the two H_1 and H_2. Finally, assume that $\ell = 1$. Then again 27 other vertices can belong to T. Every pair of disjoint vertices from this set of 27 vertices excludes one vertex that has empty intersection with both sets. Therefore, at most 18 vertices can lie in T besides the vertices of the unique K_3, so at most 21 in total. Since $n \geq 9$ we again conclude that $|T| \leq \binom{n-1}{2}$.

Case 3: $n(H) = 2$.
We may without loss of generality assume that $H = \{\{1,2,3\},\{4,5,6\}\}$ is a component of $K(n,3)[T]$. Every other vertex of T must have non-empty intersection with both vertices $x = \{1,2,3\}$ and $y = \{4,5,6\}$. The number of 3-subsets of $[n]$ that have exactly one element in common with each of x and y is equal to $\binom{3}{1}\binom{3}{1}(n-6)$. In addition, there exist exactly 18 3-subsets of $[n]$ that have two elements in common with one of x and y (and, of course, exactly one element with the other vertex). Hence there are precisely $9(n-4)$ vertices of $K(n,3)$ that have non-empty intersection with both x and y. If follows that $|T| \leq 2+9(n-4)$.

To further improve the last inequality, consider arbitrary pairwise different integers $a, b, c \in [n] \setminus [6]$. There are exactly 27 subsets of cardinality 3 which contain one of a, b, and c, and have non-empty intersection with x and y, they are listed in Table 1.

A			B			C		
$1,4,a$	$1,4,b$	$1,4,c$	$1,5,a$	$1,5,b$	$1,5,c$	$1,6,a$	$1,6,b$	$1,6,c$
$2,5,b$	$2,5,c$	$2,5,a$	$2,6,b$	$2,6,c$	$2,6,a$	$2,4,b$	$2,4,c$	$2,4,a$
$3,6,c$	$3,6,a$	$3,6,b$	$3,4,c$	$3,4,a$	$3,4,b$	$3,5,c$	$3,5,a$	$3,5,b$

Table 1. 3-subsets containg one of $a, b, c \in [n] \setminus [6]$ and having non-empty intersection with x and y

Consider the nine sets in part A of Table 1. Since we are in the case $n(H) = 2$, from each of the columns of part A, at most two subsets can lie in T. Moreover, if two subsets of a fixed column of part A lie in T, then at most four subsets of part A can belong to T. The same conclusion holds for parts B and C of Table 1 which in turn implies that at most 12 subsets from Table 1 can lie in T. Putting it other way, at least 15 vertices from Table 1 do **not** lie in T. Since a, b, and c are arbitrary integers from $[n] \setminus [6]$, it follows that

$$|T| \leq 2 + 9(n-4) - 15 \left\lfloor \frac{n-6}{3} \right\rfloor .$$

This implies that $|T| \leq \binom{n-1}{2}$ holds for $n \geq 12$. Finally, for $n \in \{9, 10, 11\}$ notice that selecting two sets from part A of Table 1 one can select at most 11 sets from Table 2.

$1,2,4$	$1,2,5$	$1,2,6$	$1,3,4$	$1,3,5$	$1,3,6$	$2,3,4$	$2,3,5$	$2,3,6$
$3,5,6$	$3,4,6$	$3,4,5$	$2,5,6$	$2,4,6$	$2,4,5$	$1,5,6$	$1,4,6$	$1,4,5$

Table 2. Specific 3-subsets

Thus $|T| \leq 2 + 9(n-4) - 15 - 7$ and we conclude that $|T| \leq \binom{n-1}{2}$ holds also for $n \in \{9, 10, 11\}$.

Case 4: $n(H) = 1$.
In this case T is an independent set, hence $|T| \leq \binom{n-1}{2}$ holds by the Erdős-Ko-Rado theorem.

We have thus proved that $\mathrm{gp}(K(n,3)) \leq \binom{n-1}{2}$ holds for every $n \geq 9$. On the other hand, $\alpha(K(n,3)) = \binom{n-1}{2}$. By Theorem 2.1 we have $\mathrm{diam}(K(n,3)) \leq 3$ which implies that every independent set of $K(n,3)$ is a general position set. Therefore, $\mathrm{gp}(K(n,3)) \geq \binom{n-1}{2}$. ∎

To conclude the section we add (while preparing the revised version) that very recently more general developments on the gp-number of Kneser graphs were reported in [18].

3. Cartesian products

In this section we prove a general lower bound on the gp-number of Cartesian product graphs. The bound is sharp as follows from the exact gp-number of the Cartesian product of two complete graphs.

The *Cartesian product* $G \,\square\, H$ of graphs G and H has the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and the edge set $E(G \,\square\, H) = \{(g,h)(g',h') : gg' \in E(G) \text{ and } h = h', \text{ or}, g = g' \text{ and } hh' \in E(H)\}$. If $(g,h) \in V(G \,\square\, H)$, then the *$G$-layer* G^h through the vertex (g,h) is the subgraph of $G \,\square\, H$ induced by the vertices $\{(g',h) : g' \in V(G)\}$. Similarly, the *H-layer* ${}^gH$ through (g,h) is the subgraph of $G \,\square\, H$ induced by the vertices $\{(g,h') : h' \in V(H)\}$. It is well-known that for given vertices $u = (g_1,h_1)$ and $v = (g_2,h_2)$ of $G \,\square\, H$ we have $\mathrm{d}_{G \square H}(u,v) = \mathrm{d}_G(g_1,g_2) + \mathrm{d}_H(h_1,h_2)$. For more on the Cartesian product see the book [8].

The announced lower bound reads as follows.

Theorem 3.1. *If G and H are connected graphs, then*

$$\mathrm{gp}(G \,\square\, H) \geq \mathrm{gp}(G) + \mathrm{gp}(H) - 2\,.$$

Proof. Let $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ be gp-sets of G and H, respectively. Let $g \in S_G$ and $h \in S_H$. We claim that

$$S = ((S_G \times \{h\}) \cup (\{g\} \times S_H)) \setminus \{(g,h)\}$$

is a general position set in $G \,\Box\, H$.

Let $u, v \in S$. Suppose first that u and v lie in the layer G^h. Since layers in Cartesian products are convex, it follows that an arbitrary shortest u,v-path P_{uv} lies completely in G^h. Since G^h is isomorphic to G, it follows that $V(P_{uv}) \cap S = \{u, v\}$. Hence $(S_G \times \{h\}) \setminus \{(g,h)\}$ is a general position set in $G \,\Box\, H$. Analogously, $(\{g\} \times S_H) \setminus \{(g,h)\}$ is a general position set.

Suppose now that $u = (g', h) \in G^h$, $v = (g, h') \in {}^gH$, and let P_{uv} be a shortest u,v-path in $G \,\Box\, H$. Suppose on the contrary that P_{uv} contains some vertex w of S different from u and v. We may without loss of generality assume that $w = (g'', h)$. Clearly, $g'' \neq g'$. Furthermore, since $(g,h) \notin S$, we also have $g'' \neq g$. Since the projection P' of P_{uv} on G^h is a shortest path between $u = (g', h)$ and (g, h) we infer that P' passes through the vertex (g'', h). This in turn implies that there exists a shortest g', g-path in G that contains g''. This is a contradiction since g, g', and g'' are pairwise different vertices.

We have thus proved that S is a general position set. Since $|S| = |S_G| + |S_H| - 2 = \mathrm{gp}(G) + \mathrm{gp}(H) - 2$ we are done. ■

The bound of Theorem 3.1 is sharp as demonstrated by the equality case of the following result.

Theorem 3.2. *If $k \geq 2$ and $n_1, \ldots, n_k \geq 2$, then*

$$\mathrm{gp}(K_{n_1} \,\Box\, \cdots \,\Box\, K_{n_k}) \geq n_1 + \cdots + n_k - k\,.$$

Moreover, $\mathrm{gp}(K_{n_1} \,\Box\, K_{n_2}) = n_1 + n_2 - 2$.

Proof. To simplify the notation set $G = K_{n_1} \,\Box\, \cdots \,\Box\, K_{n_k}$. Let further $V(K_n) = [n]$, so that $V(G) = \{(j_1, \ldots, j_k) :\ j_i \in [n_i], i \in [k]\}$.

For $i \in [k]$ set $X_i = \{(1, \ldots, 1, j, 1, \ldots, 1) :\ j \in \{2, \ldots, n_i\}\}$, where j is in the $i^{\rm th}$ coordinate. Clearly, $|X_i| = n_i - 1$. We claim that $X = \cup_{i \in [k]} X_i$ is a general position set of G.

Let u, v, and w be pairwise different vertices of X, and let $u \in X_p$, $v \in X_q$, and $w \in X_r$. If $p = q = r$, then u, v, and w are in the same K_{n_p}-layer and thus induce a triangle. So they are in a general position. Suppose next that $p = q \neq r$. Then $d_G(u,v) = 1$, $d_G(u,w) = 2$, and $d_G(v,w) = 2$, hence these three vertices are again in a general position in G. Finally, if $p \neq q \neq r$, then $d_G(u,v) = d_G(u,w) = d_G(v,w) = 2$, and we have the same conclusion. This proves the claim.

Since X is a general position set and, clearly, $|X| = \sum_{i \in k} |X_i| = n_1 + \cdots + n_k - k$, the lower bound is proved.

Let now $k = 2$, so that $G = K_{n_1} \,\square\, K_{n_2}$ and $V(G) = \{(i,j) :\ i \in [n_1], j \in [n_2]\}$. Since $\mathrm{diam}(G) = 2$, Theorem 1.1 applies. Clearly, $\omega(G) = \max\{n_1, n_2\}$.

In the rest we are going to prove that $\eta(G) = n_1 + n_2 - 2$. We will prove this assertion by induction on $n_1 + n_2$, the basic case $n_1 = n_2 = 2$ being clear. Note also that if $n_2 = 2$ and $n_1 \geq 3$, then the result also holds, that is, $\eta(G) = n_1$ in this case.

Let H be a complete multipartite subgraph of $\overline{G}$ and let $X_1, \ldots, X_k$ be the partite sets of H. We first claim that each X_i is a subset of the vertex set of some layer. If $|X_1| = 1$ there is nothing to prove. Hence let $|X_1| \geq 2$ and suppose without loss of generality that $(1,1) \in X_1$. Since X_1 is an independent set, we have $(\{2, \ldots, n_1\} \times \{2, \ldots, n_2\}) \cap X_1 = \emptyset$. We may further suppose without loss of generality that X_1 contains another vertex from $K_{n_1}^1$, say $(i,1)$. Since $(i,1)$ is adjacent to all the vertices from $\{1\} \times \{2, \ldots, n_2\}$, we conclude that $X_1 \subseteq V(K_{n_1}^1)$. This proves the claim, that is, each X_i is a subset of the vertex set of some layer.

By the claim above we may without loss of generality assume that $X_1 = \{(1,1), \ldots, (r,1)\}$, where $r \in [n_1]$. We now distinguish the following cases.

Case 1: $r = n_1$.
In this case H consists of a single complete component, that is, $k = 1$. Hence $n(H) = n_1$ and since $n_2 \geq 2$, we infer that $n(H) \leq n_1 + n_2 - 2$.

Case 2: $r < n_1$.
In this case none of the vertices from $(\{1, \ldots, r\} \times \{2, \ldots, n_2\}) \cup (\{r+1, \ldots, n_1\} \times \{1\})$ lies in H. If follows that $X_2, \ldots, X_k$ lie in the subgraph induced by $\{r+1, \ldots, n_1\} \times \{2, \ldots, n_2\}$. The latter subgraph is isomorphic to $K_{n_1-r} \,\square\, K_{n_2-1}$.

Case 2.1: $n_1 - r \geq 2$ and $n_2 - 1 \geq 2$.
In this subcase the induction hypothesis implies that

$$\eta(K_{n_1-r} \,\square\, K_{n_2-1}) = (n_1 - r) + (n_2 - 1) - 2 = n_1 + n_2 - r - 3\,.$$

It follows that

$$n(H) \leq (n_1 + n_2 - r - 3) + r = n_1 + n_2 - 3\,.$$

Case 2.2: $n_1 - r \leq 1$.
In this subcase we have $n_1 - r = 1$ and $k = 2$. Then $X_2 \subseteq \{(n_1, 2), \ldots, (n_1, n_2)\}$. Moreover, the set $\{(1,1), \ldots, (n_1 - 1, 1)\} \cup \{(n_1, 2), \ldots, (n_1, n_2)\}$ induces a complete bipartite graph of $\overline{G}$ which is of order $(n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2$.

Case 2.3: $n_2 - 1 \leq 1$.
This means that $n_2 \leq 2$, and so $n_2 = 2$, the case that was already considered.

In all the above cases we have thus proved that a complete multipartite subgraph of $\overline{G}$ is of order at most $n_1 + n_2 - 2$. Moreover, in Case 2.2 we have also found a complete multipartite subgraph of $\overline{G}$ of order exactly $n_1 + n_2 - 2$. We can conclude that $\eta(G) = n_1 + n_2 - 2$. ■

Note that the lower bound of Theorem 3.2 for at least three factors is stronger than the bound one can deduce by induction from Theorem 3.1. However, as recently proved in [10] by a probabilistic argument, the bound of Theorem 3.2 becomes very non-sharp as k grows.

4. The gp-number of some graph operations

In this section we consider the gp-number of joins of graphs, of coronas over graphs, and of line graphs. For this sake the following concept will be useful. Complete subgraphs Q and Q' in a graph G are *independent* if $d_G(u, u') \geq 2$ for every $u \in V(Q)$ and every $u' \in V(Q')$. (This concept has been very recently introduced and applied in [4].) Note that the complete subgraphs from Theorem 1.2 are independent by definition. Setting $\rho(G)$ to denote the maximum number of vertices in a union of pairwise independent complete subgraphs of G, we have:

Theorem 4.1. *If* $\mathrm{diam}(G) \in \{1, 2\}$, *then* $\mathrm{gp}(G) = \rho(G)$.

Proof. The assertion is clear if $\mathrm{diam}(G) = 1$, that is, if G is a complete graph.

Let G be a graph of diameter 2. Clearly, $\rho(G) \geq \omega(G)$, and $\rho(G) \geq \eta(G)$. Theorem 1.1 thus implies that $\rho(G) \geq \mathrm{gp}(G)$. Conversely, the $\rho(G)$ vertices from a largest union of pairwise independent cliques form a general position set by Theorem 1.2. Therefore, $\mathrm{gp}(G) \geq \rho(G)$. ■

The reason that in Theorem 4.1 $\mathrm{gp}(G)$ is expressed only with $\rho(G)$, while in Theorem 1.1 two invariants are used, is that $\rho(G)$ encapsulates $\omega(G)$ while $\eta(G)$ does not.

4.1. Joins and coronas

If G and H are disjoint graphs, then the *join* $G + H$ of G and H is the graph with the vertex set $V(G + H) = V(G) \cup V(H)$, and the edge set $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. If both G and H are complete, so it is $G + H$, and hence $\mathrm{gp}(G + H) = \mathrm{gp}(K_{n(G)} + K_{n(H)}) = \mathrm{gp}(K_{n(G)+n(H)}) = n(G + H)$. Otherwise, that is, if at least one of G and H is not complete, then $\mathrm{diam}(G + H) = 2$. In this case we have:

Proposition 4.2. *If G and H are graphs, then*

$$\begin{aligned}\mathrm{gp}(G+H) &= \max\{\omega(G)+\omega(H),\eta(G),\eta(H)\}\\ &= \max\{\omega(G)+\omega(H),\rho(G),\rho(H)\}\,.\end{aligned}$$

Proof. Since $\mathrm{diam}(G+H)=2$, Theorem 1.1 applies. It is straightforward that $\omega(G+H)=\omega(G)+\omega(H)$, and that $\eta(G+H)=\max\{\eta(G),\eta(H)\}$. Hence the first equality.

A complete subgraph Q of $G+H$ lies completely in G, or completely in H or is a join of a complete subgraph of G, and a complete subgraph of H. If Q is of the latter form, then it is at distance 1 to every other complete subgraph of $G+H$. If follows that $\rho(G+H)=\max\{\omega(G)+\omega(H),\rho(G),\rho(H)\}$. The second equality then follows by Theorem 4.1. ■

Let G and H be graphs where $V(G)=\{v_1,\ldots,v_{n(G)}\}$. The *corona* $G\circ H$ of graphs G and H is obtained from the disjoint union of G, and $n(G)$ disjoint copies of H, say $H_1,\ldots,H_{n(G)}$, where for all $i\in[n(G)]$, the vertex $v_i\in V(G)$ is adjacent to each vertex of H_i.

Theorem 4.3. *If G is a connected graph with $n(G)\ge 2$, and H is a graph, then*

$$\mathrm{gp}(G\circ H)=n(G)\rho(H)\,.$$

Proof. Let $V(G)=\{v_1,\ldots,v_{n(G)}\}$, and let $H_1,\ldots,H_{n(G)}$ be the corresponding copies of H in $G\circ H$. Note first that the statement is clear for the corona $K_2\circ K_1=P_4$. So we may assume in the rest that if $n(G)=2$, then $n(H)\ge 2$.

Let S be a gp-set of $G\circ H$. Suppose first that $S\cap V(G)\neq\emptyset$. We may assume without loss of generality that $v_1\in S$. If there exists a vertex $w\in S\cap V(H_1)$, $w\neq v_1$, then for any vertex $x\in V(G\circ H)\setminus(V(H_1)\cup\{v_1\})$, the vertex v_1 lies on a shortest w,x-path. Consequently, $S\subseteq V(H_1)\cup\{v_1\}$. Suppose that $n(G)=2$. If $x\in V(H_1)$ and $y\in V(H_2)$, then $d_{G\circ H}(x,y)=3$. It follows that the union of a general position set of H_1 and a general position set of H_2 is a general position set of $G\circ H$. But then the union of a gp-set of H_1 and a gp-set of H_2 has cardinality bigger that S because $\mathrm{gp}(H)\ge 2$, and if $n(G)\ge 3$, then we get a similar contradiction. It follows that if $v_1\in S$, then $S\cap V(H_1)=\emptyset$. But then $S'=S\cup\{w\}\setminus\{v_1\}$, where w, is an arbitrary vertex of H_1 is also a gp-set. In summary, we have proved that we may without loss of generality assume that $S\cap V(G)=\emptyset$.

So let now S be a gp-set of $G\circ H$ with $S\cap V(G)=\emptyset$. By Theorem 1.2, the components of $(G\circ H)[S]$ are independent complete graphs. Let H_i' be the subgraph of $G\circ H$ induced by the vertices from $V(H_i)\cup\{v_i\}$. Since $\mathrm{diam}(H_i')\le 2$, Theorem 4.1 implies that S restricted to H_i has at most $\rho(H)$ vertices. On the other hand, since independent complete subgraphs of H_i are pairwise at distance

2, they form (in view of Theorem 1.2) a general position set. But then taking such complete subgraphs in every H_i yields a general position set of order $n(G)\rho(H)$. ■

4.2. Line graphs of complete graphs

If G is a graph, then the *line graph* $L(G)$ of G is the graph with $V(L(G)) = E(G)$, two different vertices of $L(G)$ being adjacent if the corresponding edges share a vertex in G.

Theorem 4.4. *If $n \geq 3$, then*

$$\mathrm{gp}(L(K_n)) = \begin{cases} n; & 3 \mid n\,, \\ n-1; & 3 \nmid n\,. \end{cases}$$

Proof. Let $n \geq 3$ and $V(K_n) = [n]$. To simplify the notation set $G_n = L(K_n)$. Since $\omega(G_n) = n-1$, we have $\mathrm{gp}(G_n) \geq n-1$.

We next claim that $\mathrm{gp}(T(n)) \leq n$. Let S be a gp-set of G_n and let $K_{n_1}, \ldots, K_{n_k}$ be the connected components of $G_n[S]$, so that $\mathrm{gp}(G_n) = |S| = n_1 + \cdots + n_k$. A vertex u of G_n corresponds to an edge of K_n, that is, to a pair of vertices $\{j, j'\}$ and we may identify u with $\{j, j'\}$. Using this convention, for $i \in [k]$ set

$$X_i = \bigcup_{\{j,j'\} \in V(K_{n_i})} \{j, j'\}\,.$$

Since the complete subgraphs K_{n_i} are pairwise independent, it follows that if $i \neq i'$, then $X_i \cap X_{i'} = \emptyset$. Setting $x_i = |X_i|$ we infer that $x_i \geq n_i$ and hence

$$\mathrm{gp}(G_n) = |S| = n_1 + \cdots + n_k \leq x_1 + \cdots + x_k \leq n\,, \tag{1}$$

and the claim is proved.

If $3 \mid n$, then

$$S = \{\{3i+1, 3i+2\}, \{3i+1, 3i+3\}, \{3i+2, 3i+3\} :\ 0 \leq i \leq \frac{n}{3} - 1\}$$

is a gp-set of G_n, and hence $\mathrm{gp}(G_n) = n$.

Suppose now that $3 \nmid n$. Then at least one $n_i \neq 3$ and for it we have $n_i < x_i$. In view of (1) this means that $\mathrm{gp}(G_n) < n$. As we have already observed that $\mathrm{gp}(G_n) \geq n-1$, the argument is complete. ■

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